

A REVIEW ON MATHEMATICAL ANALYSIS FOR NEMATIC AND SMECTIC-A LIQUID CRYSTAL MODELS*

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Abstract

We review the mathematical analysis of some uniaxial, liquid crystal phases. First, we state the models for the two different studied phases: nematic and smectic-A liquid crystals. The spatial and temporal profiles of the liquid crystal configurations will be described by means of strongly nonlinear parabolic partial differential systems, which are presented at the same time. Then, we will state some results about existence, regularity, time-periodicity and stability of solutions at infinite time for both models.

It is our aim to show that, although nematic and smectic-A phases have different physical properties and are modeled by different nonlinear parabolic problems, there exists a common mathematical machinery to rewrite the models and to obtain the analytical results.

Keywords: Liquid crystals, nematic phase, smectic-A phase, Navier-Stokes equations, Ginzburg-Landau penalization, global in time solutions, time-periodic solutions, regularity, stability.

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1 Introduction

Liquid crystals (LCs) are substances which exhibit an intermediate phase of matter that has properties between those of a conventional liquid and those of a solid crystal. For instance, a LC may flow like a liquid, but its molecules may be oriented in a crystal-like way, see Figure 1. On the other hand, they have (anisotropic) optical and electro-magnetics characteristics like a solid. There are many different types of LC phases (mesophases), which can be distinguished by their different optical properties. The local average order of molecules

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thermotropic liquid crystals

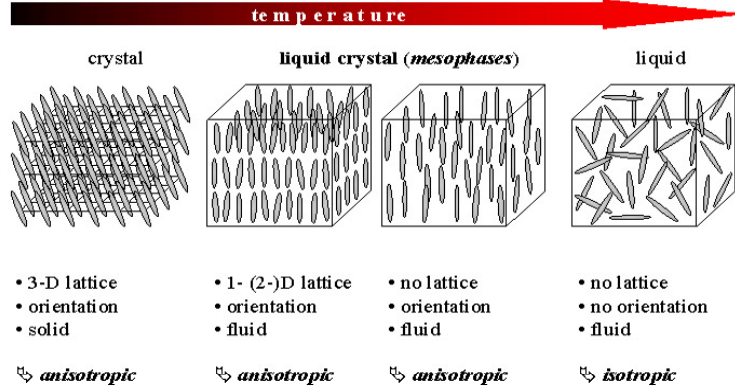


Figure 1: [<http://moebius.physik.tu-berlin.de/lc/lcs.html>'06]

characterize the LC phases, see Figure 1. Some of them are nematic or smectic phases. In the nematic phase molecules have no positional order, but they have long-range orientational order, the molecules flow and their center of mass positions are randomly distributed as in a liquid, but they point towards a preferred direction (in a local average). In the smectic phase, we can distinguish positional order (molecules form well-defined layers) and orientational order. There are different smectic phases. In the Smectic-A phase, the molecules are oriented perpendicularly to the layers, while in the Smectic-C phase, they are tilted away from the normal directions to the layers, see Figure 2. Historically, the first smectic phase observed

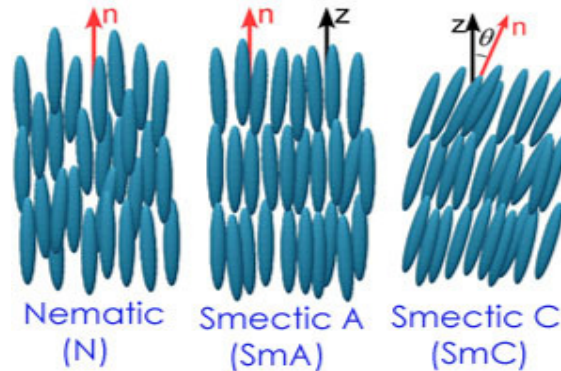


Figure 2: [<http://atom.physics.calpoly.edu>'09]

was the cholesteric phase, which exhibits a twisting of the molecules perpendicular to the director, see Figure 3. However, this type of phase will not be analyzed in our review.

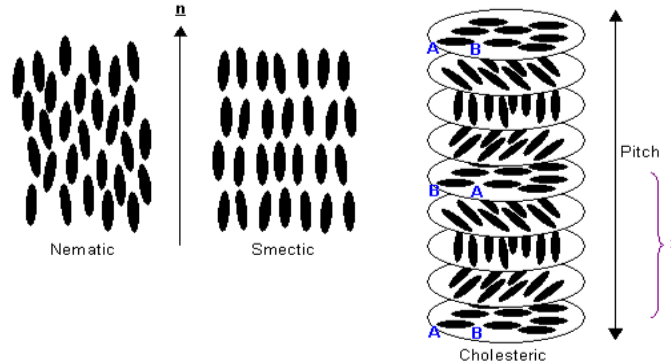


Figure 3: [<http://www.doitpoms.ac.uk>'09]

For more information about liquid crystals, see [de Gennes, Prost'93] or [Collings'02].

We will focus on nematic (**N**) and smectic-A (**SmA**) phases from a mathematical point of view, analyzing some initial-boundary or periodic-time nonlinear parabolic systems modeling these phases. We do not actually make an attempt to signpost the physical implications or applications of these problems. For this, the reader can see, for instance, [Stewart'04] in the nematic case or [Stewart'07] in the smectic-A case.

The main results presented here are the existence and uniqueness of global in time weak and regular solutions of initial-boundary or time-periodic problems, as well as the convergence of the trajectories of strong solutions to equilibrium solutions. Moreover, in this review we give some improvements for the nematic case and some simplifications for the smectic-A case, with regard to the results given previously in the literature.

Our aim is to show that, although nematic and smectic-A phases are modeled by different nonlinear parabolic problems, there exists a common mathematical machinery to obtain all the analytical results presented in this paper.

1.1 The Models

A simplified model from the original Ericksen-Leslie equations in the continuum theory of nematic LC, due to Ericksen in [Ericksen'61, Ericksen'87] and Leslie in [Leslie'68, Leslie'79], was introduced by Lin in [Lin'89] and studied (from a mathematical point of view) by Lin and Liu in [Lin,Liu'95, Lin,Liu'00] and by Coutand and Shkoller in [Coutand,Shkoller'01]. A model for Smectic-A LC was proposed by E in [E'97] and studied analytically by Liu [Liu'00].

We assume a liquid crystal confined in an open bounded domain $\Omega \subset \mathbb{R}^3$ with (regular) boundary $\partial\Omega$, which is thermally isolated during the time interval $[0, +\infty)$. Then, the

dynamic can be described by the velocity-pressure variables $\mathbf{u} : \Omega \times [0, +\infty) \mapsto \mathbb{R}^3$ and $p : \Omega \times [0, +\infty) \mapsto \mathbb{R}$ respectively. For isotropic fluids, these variables are governed by the Navier-Stokes equations, but in LC the anisotropic configurations modify the dynamic, and reciprocally, the movement has an influence on the orientation of the molecules.

The following variables can be considered in LC models:

- The so-called *director* is a unitary vectorial function $\mathbf{d} : \Omega \times [0, +\infty) \mapsto \mathbb{R}^3$, with $|\mathbf{d}| = 1$, modeling an average of the orientation of the molecules (this vector in Figure 2 is called \mathbf{n}). Owing to the head-to-tail symmetry [Collings'02], equations must be invariant changing \mathbf{d} by $-\mathbf{d}$.
- In smectic LC, the vectorial function $\mathbf{n} : \Omega \times [0, +\infty) \mapsto \mathbb{R}^3$ pointing to the single optical axis (in some monographs, this vector is called \mathbf{a} and in Figure 2 is called \mathbf{z}) is perpendicular to the layers. It is usual to impose the assumption $\nabla \times \mathbf{n} = 0$ in order to have a potential function $\varphi : \Omega \times [0, +\infty) \mapsto \mathbb{R}$, called the layer variable, such that $\mathbf{n} = \nabla \varphi$ and the level sets of φ indicate the layer structure.
- Moreover, in smectic-A LC the preferential direction of molecules is perpendicular to layers, hence \mathbf{d} is proportional to \mathbf{n} . Also, it is assumed $|\mathbf{n}| = 1$ (and therefore $\mathbf{n} = \mathbf{d}$), i.e. $|\nabla \varphi| = 1$, because the layers are incompressible.

The static equilibria are related to the (elastic) *Oseen-Frank energy*, which in the more simple case (of equal elasticity constants) can be reduced [de Gennes, Prost'93] to the convex functional $\int_{\Omega} |\nabla \mathbf{d}|^2$ (called *Dirichlet energy*).

In uniaxial nematic LC phases and under certain circumstances, the energy minimizers of the Oseen-Frank functional approximate the Landau-de Gennes minimizers [Majumdar,Zarnescu'10]. But, this might not always be the case and indeed, it can be significant differences between the classical Oseen-Frank theory and the Landau-de Gennes theory, see [Ball,Zarnescu'11]. See also [Lin,Liu'01] for a review on the static and dynamic theory of nematic and smectic-A LC, with some connections between Oseen-Frank and Landau-de Gennes theories.

To minimize the Oseen-Frank functional, the non-convex constraint $|\mathbf{d}| = 1$ is considered. In order to avoid this constraint, one possibility is to consider an approximation of the Oseen-Frank functional by a penalization functional of Ginzburg-Landau type [Bethuel et al.'93]. Indeed, by using the function

$$\mathbf{f}(\mathbf{d}) = \frac{1}{\varepsilon^2}(|\mathbf{d}|^2 - 1)\mathbf{d}$$

and the corresponding potential function

$$F(\mathbf{d}) = \frac{1}{4\varepsilon^2}(|\mathbf{d}|^2 - 1)^2 \quad (\text{i.e. } \mathbf{f}(\mathbf{d}) = \nabla_{\mathbf{d}} F(\mathbf{d})),$$

the minimization problem under the constraint $|\mathbf{d}| = 1$ is replaced by a minimization problem without constraints but applied to the non-convex penalized elastic energy:

$$(\mathbf{N}) \quad \int_{\Omega} \mathcal{E}_e(\mathbf{d}) d\mathbf{x} = \int_{\Omega} \left(\frac{1}{2} |\nabla \mathbf{d}|^2 + F(\mathbf{d}) \right) d\mathbf{x}, \quad (1)$$

$$(\mathbf{SmA}) \quad \int_{\Omega} \mathcal{E}_e(\varphi) d\mathbf{x} = \int_{\Omega} \left(\frac{1}{2} |\Delta \varphi|^2 + F(\nabla \varphi) \right) d\mathbf{x}. \quad (2)$$

It should be noticed that the balance between the Oseen-Frank energy and the penalized part is realized by the penalization parameter $\varepsilon > 0$, which appears in F .

Then, the associated Euler-Lagrange systems (i.e. the critical point equations related to each functional given in (1) or (2)) are:

$$(\mathbf{N}) \quad \boldsymbol{\omega}(\mathbf{d}) \equiv -\Delta \mathbf{d} + \mathbf{f}(\mathbf{d}) = 0, \quad (\mathbf{SmA}) \quad \omega(\varphi) \equiv \Delta^2 \varphi - \nabla \cdot \mathbf{f}(\nabla \varphi) = 0. \quad (3)$$

Note that $\boldsymbol{\omega}(\mathbf{d}) \in \mathbb{R}^3$ and $\omega(\varphi) \in \mathbb{R}$.

Now, we will introduce equations governing the dynamics of LC. The conservation of angular momentum is related to the proportionality between the material time derivative of the order parameter (\mathbf{d} in (\mathbf{N}) or φ in (\mathbf{SmA})) and the Euler-Lagrange equations given by $\boldsymbol{\omega}(\mathbf{d})$ in (\mathbf{N}) or $\omega(\varphi)$ in (\mathbf{SmA}) (see [Lin'89] for (\mathbf{N}) or [E'97] for (\mathbf{SmA})). It writes as the following equations:

$$(\mathbf{N}) \quad \partial_t \mathbf{d} + (\mathbf{u} \cdot \nabla) \mathbf{d} + \gamma \boldsymbol{\omega}(\mathbf{d}) = 0, \quad (\mathbf{SmA}) \quad \partial_t \varphi + \mathbf{u} \cdot \nabla \varphi + \gamma \omega(\varphi) = 0 \quad (4)$$

in $(0, T) \times \Omega$, where the positive proportionality constant γ is an elastic relaxation time. In [Climent, Guillen'12], we study a smectic-A model written in the vectorial variable \mathbf{n} (or \mathbf{d}), using a time material derivative with respect to \mathbf{n} and not considering φ .

The conservation of linear momentum and the incompressibility of the fluid (assuming constant density, $\rho_0 = 1$) are written as:

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla \cdot (\underline{\sigma}^d + \lambda \underline{\sigma}^e) + \nabla p = 0, \quad \nabla \cdot \mathbf{u} = 0, \quad \text{in } (0, T) \times \Omega, \quad (5)$$

where the shear stress tensor has been split, in a dissipative tensor $\underline{\sigma}^d$ plus an elastic tensor $\underline{\sigma}^e$ multiplied by an elastic constant, $\lambda > 0$. For instance, we will take the simplified tensors given in [Lin'89] for (\mathbf{N}) , and the more general tensors given in [E'97], for (\mathbf{SmA}) :

$$(\mathbf{N}) \quad \underline{\sigma}^d = \mu_4 D(\mathbf{u}), \quad \underline{\sigma}^e = -(\nabla \mathbf{d})^t \nabla \mathbf{d} \quad (6)$$

$$(\mathbf{SmA}) \quad \begin{cases} \underline{\sigma}^d = \mu_1 (\mathbf{n}^t D(\mathbf{u}) \mathbf{n}) \mathbf{n} \otimes \mathbf{n} + \mu_4 D(\mathbf{u}) + \mu_5 (D(\mathbf{u}) \mathbf{n} \otimes \mathbf{n} + \mathbf{n} \otimes D(\mathbf{u}) \mathbf{n}), \\ \underline{\sigma}^e = -\mathbf{f}(\mathbf{n}) \otimes \mathbf{n} + \nabla (\nabla \cdot \mathbf{n}) \otimes \mathbf{n} - (\nabla \cdot \mathbf{n}) \nabla \mathbf{n}. \end{cases} \quad (7)$$

Here, $\mu_1 \geq 0$, $\mu_4 > 0$, $\mu_5 \geq 0$ are dissipative constant coefficients, $D(\mathbf{u}) = (\nabla \mathbf{u} + (\nabla \mathbf{u})^t)/2$ denotes the deformation rate tensor (symmetrized velocity gradient) and $(\mathbf{a} \otimes \mathbf{b})_{i,j} = a_i b_j$ is

the tensorial product. We are dealing with a coupling system in which $\underline{\sigma}^e$ depends on \mathbf{d} in **(N)** (or \mathbf{n} in **(SmA)**) and \mathbf{u} appears in the convection term $\mathbf{u} \cdot \nabla \mathbf{d}$ in **(N)** (or $\mathbf{u} \cdot \nabla \varphi$ in **(SmA)**)).

It should be noticed that, a simplified version of the dissipative tensor has been considered in **(N)** only by simplicity, because the analytical results of the posed problems can be extended to more general tensors [Lin,Liu'00].

Then, the models consist of (3), (4), (5) and (6) for **(N)** or (7) for **(SmA)**, completed with the (Dirichlet) boundary conditions:

$$\textbf{(N)} \quad \mathbf{u}|_{\partial\Omega} = 0, \quad \mathbf{d}|_{\partial\Omega} = \mathbf{h} \quad \text{on } \partial\Omega \times (0, T) \quad (8)$$

$$\textbf{(SmA)} \quad \mathbf{u}|_{\partial\Omega} = 0, \quad \varphi|_{\partial\Omega} = \varphi_1, \quad \partial_n \varphi|_{\partial\Omega} = \varphi_2 \quad \text{on } \partial\Omega \times (0, T) \quad (9)$$

where $T > 0$ is a given final time ($T < +\infty$ or $T = +\infty$), and

- either the initial conditions:

$$\textbf{(N)} \quad \mathbf{u}|_{t=0} = \mathbf{u}_0 \quad \mathbf{d}|_{t=0} = \mathbf{d}_0 \quad \text{in } \Omega \quad (10)$$

$$\textbf{(SmA)} \quad \mathbf{u}|_{t=0} = \mathbf{u}_0 \quad \varphi|_{t=0} = \varphi_0 \quad \text{in } \Omega \quad (11)$$

- or the time-periodic conditions (fixed $T < +\infty$):

$$\textbf{(N)} \quad \mathbf{u}|_{t=0} = \mathbf{u}|_{t=T}, \quad \mathbf{d}|_{t=0} = \mathbf{d}|_{t=T} \quad \text{in } \Omega \quad (12)$$

$$\textbf{(SmA)} \quad \mathbf{u}|_{t=0} = \mathbf{u}|_{t=T}, \quad \varphi|_{t=0} = \varphi|_{t=T} \quad \text{in } \Omega. \quad (13)$$

In general, we consider time-depending boundary data \mathbf{h} or φ_1, φ_2 . In the first case, (10) or (11), the compatibility condition $\mathbf{d}_0|_{\partial\Omega} = \mathbf{h}(0)$ or $\varphi_0|_{\partial\Omega} = \varphi_1(0)$ must be assumed. In the last case, (12) or (13), it is assumed

$$\mathbf{h}|_{t=0} = \mathbf{h}|_{t=T} \quad \text{or} \quad \varphi_1|_{t=0} = \varphi_1|_{t=T} \quad \text{and} \quad \varphi_2|_{t=0} = \varphi_2|_{t=T} \quad \text{in } \Omega. \quad (14)$$

Finally, all the mathematical results that we will present in this paper are dependent of the penalization parameter ε . Up to known, there are very few results taking limits as $\varepsilon \rightarrow 0$.

1.1.1 Reformulation of Nematic Model

Taking into account that

$$\nabla \cdot ((\nabla \mathbf{d})^t \nabla \mathbf{d}) = \nabla \left(\frac{|\nabla \mathbf{d}|^2}{2} + F(\mathbf{d}) \right) + (\nabla \mathbf{d})^t (\Delta \mathbf{d} - \mathbf{f}(\mathbf{d})) = \nabla \mathcal{E}_e(\mathbf{d}) - (\nabla \mathbf{d})^t \boldsymbol{\omega}(\mathbf{d})$$

and $\nabla \cdot (\mu_4 D(\mathbf{u})) = \frac{\mu_4}{2} \Delta \mathbf{u}$ (since $\nabla \cdot \mathbf{u} = 0$), (4), (5) and (6) can be rewritten as the following PDE system in $(0, T) \times \Omega$:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p - \lambda (\nabla \mathbf{d})^t \boldsymbol{\omega}(\mathbf{d}) = 0, & \nabla \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{d} + (\mathbf{u} \cdot \nabla) \mathbf{d} + \gamma \boldsymbol{\omega}(\mathbf{d}) = 0, & -\Delta \mathbf{d} + \mathbf{f}(\mathbf{d}) = \boldsymbol{\omega}(\mathbf{d}), \end{cases} \quad (15)$$

where $\nu = \mu_4/4$ and p is a reformulated potential function equal to $p + \lambda \mathcal{E}_e(\mathbf{d})$.

1.1.2 Reformulation of Smectic-A Model

By splitting the symmetric dissipative tensor into the linear and nonlinear part

$$\underline{\sigma}^d = \mu_4 D(\mathbf{u}) + \underline{\sigma}_{nl}^d(D(\mathbf{u}), \nabla \varphi),$$

where $\underline{\sigma}_{nl}^d := \mu_1(\mathbf{n}^t D(\mathbf{u}) \mathbf{n}) \mathbf{n} \otimes \mathbf{n} + \mu_5(D(\mathbf{u}) \mathbf{n} \otimes \mathbf{n} + \mathbf{n} \otimes D(\mathbf{u}) \mathbf{n})$, notice that (again for $\nu = \mu_4/2$)

$$-\nabla \cdot \underline{\sigma}^d = -\nu \Delta \mathbf{u} - \nabla \cdot \underline{\sigma}_{nl}^d.$$

On the other hand, by decomposing the penalization term in the elastic tensor $\underline{\sigma}^e$ as follows

$$\underline{\sigma}^e = -\mathbf{f}(\mathbf{n}) \otimes \mathbf{n} + \underline{\sigma}_{np}^e(\mathbf{n}),$$

where $\underline{\sigma}_{np}^e(\mathbf{n}) := \nabla(\nabla \cdot \mathbf{n}) \otimes \mathbf{n} - (\nabla \cdot \mathbf{n}) \nabla \mathbf{n}$ is the non-penalized tensor, and taking into account that

$$\nabla \cdot (\mathbf{f}(\mathbf{n}) \otimes \mathbf{n}) = (\nabla \cdot \mathbf{f}(\nabla \varphi)) \nabla \varphi + \mathbf{f}_i(\nabla \varphi) \partial_i \nabla \varphi = (\nabla \cdot \mathbf{f}(\nabla \varphi)) \nabla \varphi + \nabla F(\nabla \varphi)$$

and

$$\begin{aligned} (\nabla \cdot \underline{\sigma}_{np}^e(\mathbf{n}))_j &= (\nabla \cdot (\nabla(\nabla \cdot \mathbf{n}) \otimes \mathbf{n} - (\nabla \cdot \mathbf{n}) \nabla \mathbf{n}))_j = (\nabla \cdot (\nabla(\Delta \varphi) \otimes \nabla \varphi - \Delta \varphi \nabla^2 \varphi))_j \\ &= \partial_i (\partial_i (\Delta \varphi) \partial_j \varphi - \Delta \varphi \partial_{ij}^2 \varphi) = \Delta^2 \varphi \partial_j \varphi + \partial_i (\Delta \varphi) \partial_{ij}^2 \varphi - \partial_i (\Delta \varphi) \partial_{ij}^2 \varphi - \Delta \varphi \partial_i \partial_{ij}^2 \varphi \\ &= \Delta^2 \varphi \partial_j \varphi - \Delta \varphi \partial_j \Delta \varphi = \Delta^2 \varphi \partial_j \varphi - \frac{1}{2} \partial_j (|\Delta \varphi|^2), \end{aligned}$$

we have

$$\begin{aligned} -\nabla \cdot \underline{\sigma}^e &= (\nabla \cdot \mathbf{f}(\nabla \varphi)) \nabla \varphi + \nabla F(\nabla \varphi) - \Delta^2 \varphi \nabla \varphi + \nabla \left(\frac{|\Delta \varphi|^2}{2} \right) \\ &= (-\Delta^2 \varphi + \nabla \cdot \mathbf{f}(\nabla \varphi)) \nabla \varphi + \nabla \left(F(\nabla \varphi) + \frac{|\Delta \varphi|^2}{2} \right) = -\omega(\varphi) \nabla \varphi + \nabla \mathcal{E}_e(\varphi). \end{aligned}$$

Then, (4), (5) and (7) can be rewritten as the following PDE system in $(0, T) \times \Omega$:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} - \nabla \cdot \underline{\sigma}_{nl}^d - \lambda \omega(\varphi) \nabla \varphi + \nabla p = 0, \\ \partial_t \varphi + \mathbf{u} \cdot \nabla \varphi + \gamma \omega(\varphi) = 0, & \Delta^2 \varphi - \nabla \cdot \mathbf{f}(\nabla \varphi) = \omega(\varphi), \end{cases} \quad (16)$$

where p is a reformulated potential function equal to $p + \lambda \mathcal{E}_e(\varphi)$.

1.2 Notation

- In general, the notation will be abridged. We set $L^p = L^p(\Omega)$, $p \geq 1$, $H_0^1 = H_0^1(\Omega)$, etc. $H^{-1} = H^{-1}(\Omega)$ is the dual space of H_0^1 . If $X = X(\Omega)$ is a space of functions defined in the open set Ω , we denote by $L^p(0, T; X)$ the Banach space $L^p(0, T; X(\Omega))$. Also, boldface letters will be used for vectorial spaces, for instance $\mathbf{L}^2 = L^2(\Omega)^3$.
- The L^p norm is denoted by $|\cdot|_p$, $1 \leq p \leq \infty$, the H^m norm by $\|\cdot\|_m$ (in particular $|\cdot|_2 = \|\cdot\|_0$), $\|\cdot\|_{-1}$ denotes the usual norm in H^{-1} and the product norm in $H^n \times H^m$ by $\|\cdot\|_{n \times m}$. The inner product of $L^2(\Omega)$ is denoted by (\cdot, \cdot) .
- We will consider Ω regular enough to have the following equivalent norms:

$$\begin{aligned} \|v\|_1 &\approx |\nabla v|_2 \quad \text{in } H_0^1, & \|v\|_2 &\approx |\Delta v|_2 \quad \text{in } H_0^1 \cap H^2, \\ \|v\|_3 &\approx |\nabla(\Delta v)|_2 + |\Delta v|_2 = \|\Delta v\|_1 \quad \text{in } H_0^1 \cap H^3, & \|v\|_4 &\approx |\Delta^2 v|_2 \quad \text{in } H_0^1 \cap H^4. \end{aligned}$$

- We set \mathcal{V} the space formed by all fields $\mathbf{u} \in C_0^\infty(\Omega)^3$ satisfying $\nabla \cdot \mathbf{u} = 0$. We denote \mathbf{H} (respectively \mathbf{V}) the closure of \mathcal{V} in \mathbf{L}^2 (respectively \mathbf{H}^1). \mathbf{H} and \mathbf{V} are Hilbert spaces for the norms $|\cdot|_2$ and $\|\cdot\|_1$, respectively. Furthermore,

$$\mathbf{H} = \{\mathbf{u} \in \mathbf{L}^2; \nabla \cdot \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}, \quad \mathbf{V} = \{\mathbf{u} \in \mathbf{H}^1; \nabla \cdot \mathbf{u} = 0, \mathbf{u} = 0 \text{ on } \partial\Omega\}$$

- From now on, $C, C_i, D > 0$, for $i \geq 0$, will denote different constants, depending only on the fixed data of the problem, as Ω, ε and boundary data: \mathbf{h} or φ_1, φ_2 (and $\mathbf{u}_0, \mathbf{d}_0$ or φ_0 for the initial-value problem).

1.3 Some comments about time-independent boundary data

Assuming time-independent boundary data, an important fact of both models is their dissipative character, because they admit (at least for regular solutions) the following energy equalities (see [Lin,Liu'95] and [Liu'00] respectively):

$$\text{(N)} \quad \frac{d}{dt} \left(\frac{1}{2} |\mathbf{u}|_2^2 + \lambda \int_{\Omega} \mathcal{E}_e(\mathbf{d}) \right) + \nu |\nabla \mathbf{u}|_2^2 + \lambda \gamma |\boldsymbol{\omega}(\mathbf{d})|_2^2 = 0, \quad (17)$$

$$\text{(SmA)} \quad \frac{d}{dt} \left(\frac{1}{2} |\mathbf{u}|_2^2 + \lambda \int_{\Omega} \mathcal{E}_e(\varphi) \right) + \nu |\nabla \mathbf{u}|_2^2 + \int_{\Omega} \underline{\sigma}_{nl}^d : D(\mathbf{u}) + \lambda \gamma |\boldsymbol{\omega}(\varphi)|_2^2 = 0, \quad (18)$$

for any time $t \in (0, +\infty)$. Equality (17) is obtained from (15), by taking \mathbf{u} as test function in the \mathbf{u} -system, $\boldsymbol{\omega}(\mathbf{d})$ in the \mathbf{d} -system and adding up. The equality (18) is obtained from (16) by taking \mathbf{u} as test function in the \mathbf{u} -system, $\boldsymbol{\omega}(\varphi)$ in the φ -equation and adding up. Note that the time-independent boundary data is applied to vanish boundary integrals (arising

after integrating by parts) using that $\partial_t \mathbf{d}|_{\partial\Omega} = 0$ for **(N)** or $\partial_t \varphi|_{\partial\Omega} = 0$ and $\partial_t(\partial_n \varphi)|_{\partial\Omega} = 0$ for **(SmA)**.

Equalities (17) and (18) imply that the total free energy (that is, the kinetic energy $\frac{1}{2}|\mathbf{u}|_2^2$ plus the elastic energy $\lambda \int_{\Omega} \mathcal{E}_e(\mathbf{d})$ for **(N)** or $\lambda \int_{\Omega} \mathcal{E}_e(\varphi)$ for **(SmA)**) decreases with respect to time, with a rate proportional to the dissipative part $|\nabla \mathbf{u}|_2^2$ and $|\boldsymbol{\omega}(\mathbf{d})|_2^2$ for **(N)** or $|\omega(\varphi)|_2^2$ for **(SmA)**.

Certainly, the existence of weak and strong solutions and the long-time convergence to no-flow states, for both nematic and smectic cases, are reassuring in terms of the model verification. If solutions did not behave like this, then the model would be incorrect.

On the other hand, it is important to remark that, like boundary data \mathbf{h} or φ_1, φ_2 are time-independent, the following (static) critical points are steady solutions (and in particular time-periodic solutions):

$$\begin{aligned} \text{(N)} \quad & \begin{cases} \mathbf{u} = 0, \\ \mathbf{d} : \text{ any solution of the problem: } -\Delta \mathbf{d} + \mathbf{f}(\mathbf{d}) = 0 \text{ in } \Omega, \mathbf{d} = \mathbf{h} \text{ on } \partial\Omega, \\ p = -\lambda \mathcal{E}_e(\mathbf{d}). \end{cases} \\ \text{(SmA)} \quad & \begin{cases} \mathbf{u} = 0, \\ \varphi : \text{ any solution of } \Delta^2 \varphi - \nabla \cdot \mathbf{f}(\nabla \varphi) = 0 \text{ in } \Omega, \varphi = \varphi_1, \partial_n \varphi = \varphi_2 \text{ on } \partial\Omega, \\ p = -\lambda \mathcal{E}_e(\varphi). \end{cases} \end{aligned}$$

Therefore, in order to consider nontrivial time-periodic problems, it will be essential to assume time-dependent and time-periodic boundary data (satisfying (14)). This situation occurs, for example, when an external magnetic or electric force is acting such that the molecules return to their initial position periodically. Other interesting situation is to consider time-independent boundary data but assuming a time-periodic force $g(t)$ (for instance electrical periodic impulses) acting on the system. In this case, the same results about time-periodic solutions could be obtained.

1.4 Some simplifications

- In the following, to make the exposition clearer and without loss of generality, we fix the constants of the problem, excepting the viscosity μ_4 , taking

$$\lambda = \gamma = \mu_1 = \mu_5 = 1, \quad (\text{recall } \nu = \mu_4/2).$$

- The case of time-dependent boundary data requires to introduce a lifting function, getting energy equalities like (17) and (18), where source terms depending on the time derivative of the boundary data appear, see (21) below. For clarity in the statement of

results, particularly in the smectic case, we will give the regularity hypotheses on the lifting function, from which one could obtain the hypotheses about the boundary data \mathbf{h} or φ_1 and φ_2 .

- In definitions of weak and strong solutions, we will not specify the regularity of p , $\partial_t \mathbf{u}$, $\partial_t \varphi$, which can be deduced from the other variables.

1.5 Other related problems

Nematic and smectic models can be viewed as particular cases of Navier-Stokes equations coupled with phase-field equations of Allen-Cahn type with an elastic tensor of Korteweg type. Other related problems appear for instance modeling the mixture of two incompressible fluids [Liu,Shen'03], or the effect of bending elasticity energy for the vesicle membranes in the fluid [Du et al.'07], etc.

On the other hand, in order to model biaxial LC with a Landau-de Gennes energy, coupled problems between Navier-Stokes and Q-tensor systems arise [Paicu,Zarnescu'12], which can be view again as Navier-Stokes equations coupled with tensorial phase-field equations of Allen-Cahn type.

2 Nematic Problem

The initial-boundary value problem associated to (15) with time-independent boundary data has been studied in [Lin,Liu'95], obtaining existence of weak solutions in $[0, T]$ for all $T > 0$ (by using a semi-Galerkin discretization, where the \mathbf{d} -system is hold at infinity dimension), existence and uniqueness of regular solution in $[0, +\infty)$ for dominant viscosity (taking ν big enough) and some properties at infinite time.

In the last years, we have done (jointly some collaborators) some analytical contributions to the nematic problem that will be described in the following points:

- In [Climent et al.'06] we prove existence of weak time-periodic solutions of (15). Verification of maximum principle is used in the argument. Indeed, assuming $|\mathbf{h}| \leq 1$ on $\partial\Omega \times (0, T)$ and $\mathbf{h}(0) = \mathbf{h}(T)$ on $\partial\Omega$, given $\mathbf{u} \in L^2(0, T; \mathbf{V}) \cap L^\infty(0, T; \mathbf{H})$, any weak solution $\mathbf{d} \in L^2(0, T; \mathbf{H}^1) \cap L^\infty(0, T; \mathbf{L}^2)$ of the \mathbf{d} -problem associated to (15), verifies $|\mathbf{d}(x, t)| \leq 1$ a.e. in $\Omega \times (0, T)$. This property lets us to take an equivalent problem considering the truncated potential

$$\tilde{f}(\mathbf{d}) = \begin{cases} f(\mathbf{d}) & \text{if } |\mathbf{d}| \leq 1 \\ 0 & \text{if } |\mathbf{d}| > 1 \end{cases}$$

instead of $f(\mathbf{d})$. Due to the time-dependent boundary data $\mathbf{h}(t)$, an “elliptic lifting” function (depending on an elliptic problem, see (19) below) is used. On the other hand, we introduce a fully Galerkin discretization of the problem, proving existence and uniqueness of an approximate solution associated to arbitrary initial conditions. Then, a Leray-Schauder argument applied to the operator mapping the initial with the final time allows us to prove the existence of Galerkin time-periodic solutions. Finally, a limit argument is used.

- In [Climent et al.’09], firstly, the initial-boundary problem (15) is considered, obtaining the existence of global in time (up to infinity time) weak solutions, the existence of global regular solutions for big enough viscosity coefficient, and the weak/strong uniqueness. Here, an elliptic lifting function is again used. Secondly, using these previous results and the existence of time-periodic weak solutions proved in [Climent et al.’06], the regularity of any time-periodic weak solution is deduced for large viscosity coefficient. Now, a different lifting function (based on a parabolic problem) is chosen in order to obtain H^3 -estimates for the homogeneous variable related to \mathbf{d} . This “parabolic lifting” is introduced since we need some specific energy inequalities in strong norms which not only yield uniform estimates of the solutions, but also provide estimates of the convergence rate.
- In [Climent et al.’10] we study stability and asymptotic stability properties at infinite time for a nematic crystal model with additional stretching terms, under periodic boundary conditions.

After studying the smectic-A phase with stretching terms in [Climent,Guillen’10] where the maximum principle is not verified and only the elliptic lifting function was used, we can prove all results obtained for nematic models without using either maximum principle or parabolic lifting. Also, the computations done in smectic case in [Climent,Guillen’10] will be now slightly simplified.

We define the following elliptic lifting function $\tilde{\mathbf{d}}(t)$ as the weak solution of the Laplace-Dirichlet problem

$$\begin{cases} -\Delta \tilde{\mathbf{d}} = 0 & \text{in } \Omega, \\ \tilde{\mathbf{d}} = \mathbf{h}(t) & \text{on } \partial\Omega. \end{cases} \quad (19)$$

In the time-periodic case, since by hypothesis $\mathbf{h}(0) = \mathbf{h}(T)$ on $\partial\Omega$, then $\tilde{\mathbf{d}}(0) = \tilde{\mathbf{d}}(T)$ in Ω .

Therefore, if we define $\hat{\mathbf{d}} = \mathbf{d} - \tilde{\mathbf{d}}$, then $\Delta \hat{\mathbf{d}} = \Delta \mathbf{d}$ in $\Omega \times (0, T)$ and $\hat{\mathbf{d}} = 0$ on $\partial\Omega \times (0, T)$. In the time-periodic case, $\mathbf{d}(0) = \mathbf{d}(T)$ if and only if $\hat{\mathbf{d}}(0) = \hat{\mathbf{d}}(T)$. Then, we can rewrite the

problem (16)-(8) in the variables $(\mathbf{u}, \widehat{\mathbf{d}})$ (with $\mathbf{d} = \widehat{\mathbf{d}} + \widetilde{\mathbf{d}}$) as follows:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p - (\nabla \mathbf{d})^t (-\Delta \widehat{\mathbf{d}} + \mathbf{f}(\mathbf{d})) = 0, & \nabla \cdot \mathbf{u} = 0, \\ \partial_t \widehat{\mathbf{d}} + (\mathbf{u} \cdot \nabla) \mathbf{d} - \Delta \widehat{\mathbf{d}} + \mathbf{f}(\mathbf{d}) = -\partial_t \widetilde{\mathbf{d}}, \\ \mathbf{u}|_{\partial\Omega} = 0, \quad \widehat{\mathbf{d}}|_{\partial\Omega} = 0, \end{cases} \quad (20)$$

jointly to either the initial condition $\mathbf{u}(0) = \mathbf{u}_0$, $\widehat{\mathbf{d}}(0) = \mathbf{d}_0 - \widetilde{\mathbf{d}}(0)$ or the time-periodic conditions $\mathbf{u}(0) = \mathbf{u}(T)$, $\widehat{\mathbf{d}}(0) = \widehat{\mathbf{d}}(T)$.

Then, if \mathbf{u} and \mathbf{d} are regular enough, the following differential inequality holds:

$$\frac{d}{dt} \left(\frac{1}{2} \|\mathbf{u}\|_2^2 + \frac{1}{2} \|\nabla \widehat{\mathbf{d}}\|_2^2 + \int_{\Omega} F(\mathbf{d}) \right) + \nu \|\nabla \mathbf{u}\|_2^2 + \frac{1}{2} \|\Delta \widehat{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|_2^2 \leq \frac{1}{2} \|\partial_t \widetilde{\mathbf{d}}\|_2^2 \quad (21)$$

for $t \in (0, +\infty)$, which is a modification of (17) due to the time dependent boundary data.

The weak estimates of (\mathbf{u}, \mathbf{d}) (see (23), (24), below) are obtained from (21) using the auxiliary variable $\mathbf{w} = -\Delta \widehat{\mathbf{d}} + \mathbf{f}(\mathbf{d})$ (see [Climent et al.'09]). On the other hand, to obtain the strong estimates of (\mathbf{u}, \mathbf{d}) (see (25), (26) below), we are going to use the modified auxiliary variable

$$\widehat{\mathbf{w}} := -\Delta \widehat{\mathbf{d}} + \mathbf{f}(\mathbf{d}) + \partial_t \widetilde{\mathbf{d}} = \mathbf{w} + \partial_t \widetilde{\mathbf{d}}$$

because $\widehat{\mathbf{w}}|_{\partial\Omega} = 0$. Concretely, from definition of $\widehat{\mathbf{w}}$, we have

$$\|\widehat{\mathbf{d}}\|_2 \leq \|\widehat{\mathbf{w}}\|_2 + C(\|\mathbf{d}\|_1^3 + \|\mathbf{d}\|_2) + \|\partial_t \widetilde{\mathbf{d}}\|_2, \quad \|\mathbf{d}\|_2 \leq \|\widehat{\mathbf{d}}\|_2 + \|\widetilde{\mathbf{d}}\|_2.$$

Imposing $\partial_t \widetilde{\mathbf{d}} \in L^\infty(0, +\infty; \mathbf{L}^2)$, $\widetilde{\mathbf{d}} \in L^\infty(0, +\infty; \mathbf{H}^2)$ and the weak estimate $\mathbf{d} \in L^\infty(0, +\infty; \mathbf{H}^1)$, we get

$$\|\widehat{\mathbf{d}}\|_2 \leq C(\|\widehat{\mathbf{w}}\|_2 + 1), \quad \|\mathbf{d}\|_2 \leq C(\|\widehat{\mathbf{w}}\|_2 + 1).$$

Then, taking respectively $A\mathbf{u}$ (where $A = P_{\mathbf{H}}(-\Delta)$ is the Stokes operator) and $\widehat{\mathbf{w}}$ as test functions in (20), we get

$$\begin{aligned} \frac{d}{dt} \left(\|\mathbf{u}\|_1^2 + \|\widehat{\mathbf{w}}\|_2^2 \right) + \frac{\nu}{2} \|\mathbf{u}\|_2^2 + \|\widehat{\mathbf{w}}\|_1^2 &\leq C(\|\widehat{\mathbf{w}}\|_2^2 + 1) \\ + \frac{C}{\nu} \left(\|\mathbf{u}\|_2^2 \|\mathbf{u}\|_1^2 + \|\widehat{\mathbf{w}}\|_1^2 (\|\widehat{\mathbf{w}}\|_2^2 + 1) \right) &+ \frac{C}{\nu} (\|\widehat{\mathbf{w}}\|_2^2 + 1) \end{aligned} \quad (22)$$

for $t \in (0, +\infty)$, where $C > 0$ are different constants always independent of ν . The global regularity will be obtained for large enough ν using (22), (see (28) below).

2.1 The initial-value problem up to infinite time

Definition 1 (\mathbf{u}, \mathbf{d}) is said a weak solution (in the time interval $(0, +\infty)$) of (15)-(10) if

$$\nabla \cdot \mathbf{u} = 0 \text{ in } Q, \quad \mathbf{u}|_{\Sigma} = 0, \quad \mathbf{d}|_{\Sigma} = \mathbf{h} \quad \text{a.e. } t \in (0, +\infty),$$

$$\|(\mathbf{u}(t), \mathbf{d}(t))\|_{0 \times 1} \leq C_1 \quad \forall t \geq 0 \quad \text{i.e. } (\mathbf{u}, \mathbf{d}) \in L^\infty(0, +\infty; \mathbf{L}^2 \times \mathbf{H}^1), \quad (23)$$

$$\forall \gamma > 0, \quad e^{-\gamma t} \int_0^t e^{\gamma s} \|(\mathbf{u}(s), \mathbf{d}(s))\|_{1 \times 2}^2 ds \leq C_2 \left(1 + \frac{1}{\nu}\right), \quad \forall t \geq 0, \quad (24)$$

where $C_1, C_2 > 0$ are constants independent of ν , verifying

$$\langle \partial_t \mathbf{u}, \mathbf{v} \rangle + ((\mathbf{u} \cdot \nabla) \mathbf{u}, \mathbf{v}) + (\nabla \mathbf{u}, \nabla \mathbf{v}) + (\nabla \mathbf{d}^t \Delta \mathbf{d}, \mathbf{v}) = 0 \quad \text{in } \mathcal{D}'(0, +\infty), \quad \forall \mathbf{v} \in \mathbf{V},$$

$$\partial_t \mathbf{d} + (\mathbf{u} \cdot \nabla) \mathbf{d} + \mathbf{f}(\mathbf{d}) - \Delta \mathbf{d} = 0 \quad \text{a.e. in } (0, +\infty) \times \Omega,$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{d}(0) = \mathbf{d}_0 \quad \text{in } \Omega.$$

In the case of a finite time interval $(0, T)$ (with $T < \infty$), (24) holds even when $\gamma = 0$, i.e. $(\mathbf{u}, \mathbf{d}) \in L^2(0, T; \mathbf{H}^1 \times \mathbf{H}^2)$. It is not possible to prove $\int_0^{+\infty} \|(\mathbf{u}(s), \mathbf{d}(s))\|_{1 \times 2}^2 ds \leq C_2$ because the bound of $(\mathbf{u}, \mathbf{d}) \in L^2(0, T; \mathbf{H}^1 \times \mathbf{H}^2)$ can blow up as $T \rightarrow +\infty$. In fact, (24) is a weighted estimate as $T \rightarrow +\infty$. For instance, a constant function $g(t) = C$ verifies $e^{-\gamma t} \int_0^t e^{\gamma s} g(s) ds \leq C$ is bounded but $g \notin L^\infty(0, +\infty)$.

Definition 2 A weak solution (\mathbf{u}, \mathbf{d}) of (15)-(10) in $(0, +\infty)$ is said a strong solution if

$$\|(\mathbf{u}(t), \mathbf{d}(t))\|_{1 \times 2} \leq C_3 \quad \forall t \geq 0, \quad (25)$$

$$\forall \gamma > 0, \quad e^{-\gamma t} \int_0^t e^{\gamma s} \|(\mathbf{u}(s), \mathbf{d}(s))\|_{2 \times 3}^2 ds \leq C_4, \quad \forall t \geq 0, \quad (26)$$

verifying the following system a.e. in $(0, +\infty) \times \Omega$:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = -\nabla \mathbf{d}^t \Delta \mathbf{d}, & \nabla \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{d} + (\mathbf{u} \cdot \nabla) \mathbf{d} = \Delta \mathbf{d} - \mathbf{f}(\mathbf{d}). \end{cases} \quad (27)$$

Again, for a finite time interval $(0, T)$ ($T < \infty$), $\gamma = 0$ can be taken in (26). \square

Theorem 3 (Existence of global weak solution) Let Ω, \mathbf{h} regular enough, such that lifting function $\tilde{\mathbf{d}}$ defined in (19) satisfies $\tilde{\mathbf{d}} \in L^\infty(0, +\infty; \mathbf{H}^2)$ and $\partial_t \tilde{\mathbf{d}} \in L^\infty(0, +\infty; \mathbf{L}^2)$. Assume $(\mathbf{u}_0, \mathbf{d}_0) \in \mathbf{H} \times \mathbf{H}^1$ in Ω , verifying the compatibility condition $\mathbf{d}_0|_{\partial\Omega} = \mathbf{h}(0)$. Then there exists a weak solution (\mathbf{u}, \mathbf{d}) of (15)-(10) in $(0, +\infty)$ which verifies (23)-(24).

To prove this theorem a semi-Galerkin method is used (a Galerkin approximation is considered only for velocity, whereas the vector director \mathbf{d} remains at infinite dimension). For the existence of solution in $\Omega \times (0, T)$ the lifting (19) is used jointly to inequality (21). The extension of the solution to whole time interval $(0, +\infty)$ can be done following the proof of Theorem 4.2 in [Climent, Guillen'10]. It should be noticed that it is not necessary neither $|\mathbf{h}(t)| \leq 1$ nor large enough viscosity ν .

Theorem 4 (Existence of global strong solution for ν large) *Under hypothesis of Theorem 3, if moreover, $(\mathbf{u}_0, \mathbf{d}_0) \in \mathbf{H}^1 \times \mathbf{H}^2$ with $\|(\mathbf{u}_0, \mathbf{d}_0)\|_{H^1 \times H^2} \leq M_0$ and the lifting function satisfies*

$$\tilde{\mathbf{d}} \in L^\infty(0, +\infty; \mathbf{H}^3) \quad \text{and} \quad \partial_t \tilde{\mathbf{d}} \in L^\infty(0, +\infty; \mathbf{H}^1),$$

then, there exists $\nu_0 = \nu_0(M_0, \tilde{\mathbf{d}}, \partial_t \tilde{\mathbf{d}})$ such that for each $\nu \geq \nu_0$, there exists an unique strong solution of (15)-(10) in $(0, +\infty)$, which verifies (25) and (26) with constants C_3, C_4 independent of ν .

The proof of this theorem is based on the following differential inequality:

$$\Phi' + \left(\frac{\nu}{2} - \frac{C}{\nu} \Phi_1 \right) \Psi_1 + \left(1 - \frac{C}{\nu} (\Phi_2 + 1) \right) \Psi_2 \leq D(\Phi_2 + 1), \quad t \in (0, \infty), \quad (28)$$

obtained from (22), where

$$\Phi = \Phi_1 + \Phi_2 \quad \text{with} \quad \Phi_1(t) = \|\mathbf{u}\|_1^2, \quad \Phi_2(t) = |\hat{\mathbf{w}}|_2^2,$$

$$\Psi_1(t) = \|\mathbf{u}\|_2^2, \quad \Psi_2(t) = |\nabla \hat{\mathbf{w}}|_2^2$$

and $C, D > 0$ are constants (independent of ν , for each ν separated from zero, for instance $\nu \geq 1/2$). Inequality (28) is rather similar to the corresponding inequality obtained in [Climent et al.'09], where a different lifting function (using a parabolic problem) and the maximum principle were used.

Theorem 4 is not a stability result for initial data close to equilibrium points. Here, fixed an initial regular data, no necessarily near to an equilibrium solution, if the viscous coefficient is big enough, we have unique global regular solution in $(0, +\infty)$. That means a dominant viscosity avoids the possibility of blow up in finite time of strong norms. It should also be noticed that, ν_0 is not decreasing to zero as $M_0 \rightarrow 0$, therefore, this result do not implies global in time regularity imposing only initial small data. This fact is an important difference with the Navier-Stokes problem without coupling with the vector director. By contrary, the following result is a extension of the Navier-Stokes framework.

Theorem 5 (Weak/strong uniqueness) *If $(\mathbf{u}_1, \mathbf{d}_1)$ is a weak solution of (15)-(10) and $(\mathbf{u}_2, \mathbf{d}_2)$ is a strong solution of (15)-(10), then both solutions coincide.*

This theorem proves that it is possible to ensure the uniqueness of solution if it exists, at least, a strong solution. To prove it, a classic argument of strong/weak uniqueness can be used (see for instance [Lions'96] for the Navier-Stokes case), despite the high nonlinear character of the elastic tensor.

2.2 Behavior at infinite time

In this section, we consider the initial-value problem with time-independent boundary data. The results can be seen in [Climent et al.'10], [Liu,Wu,Xu'09] and [Wu'10].

The asymptotic behavior as $t \rightarrow \infty$ is not clear. It is possible that there exists a global solution that converges to an equilibrium state, which may not necessarily be a local minimizer of the energy $\frac{1}{2}|\mathbf{u}(t)|_2^2 + \int_{\Omega} \mathcal{E}_e(d)$. Moreover, the set of equilibria states might be a continuum set (see, for example, [Haraux'91]).

Theorem 6 (Asymptotic stability) *Under conditions of Theorem 4, the total energy*

$$E(\mathbf{u}(t), \mathbf{d}(t)) = E_k(\mathbf{u}(t)) + E_e(\mathbf{d}(t)) := \frac{1}{2}|\mathbf{u}(t)|_2^2 + \int_{\Omega} \left(\frac{1}{2}|\nabla \mathbf{d}(t)|_2^2 + F(\mathbf{d}(t)) \right)$$

(sum of kinetic and elastic energies), satisfies

$$E(\mathbf{u}(t), \mathbf{d}(t)) \searrow E_{\infty} = E_e(\bar{\mathbf{d}}) \quad \text{as } t \uparrow +\infty,$$

where $\bar{\mathbf{d}}$ is a critical point of the elastic energy, that is, a solution of the stationary problem

$$\begin{cases} -\Delta \bar{\mathbf{d}} + f(\bar{\mathbf{d}}) = 0 & \text{in } \Omega, \\ \bar{\mathbf{d}}|_{\partial\Omega} = \mathbf{h}. \end{cases}$$

Moreover, the strong solution (\mathbf{u}, \mathbf{d}) satisfies

$$\mathbf{u}(t) \rightarrow 0 \text{ in } \mathbf{H}_0^1(\Omega), \quad \Delta \mathbf{d}(t) - f(\mathbf{d}(t)) \rightarrow 0 \text{ in } L^2(\Omega) \quad \text{and} \quad \mathbf{d}(t) \rightarrow \bar{\mathbf{d}} \text{ in } H^2(\Omega).$$

We only give here an sketch of the proof of this theorem, splitting it into four steps:

Step 1: We consider the energy $E(\mathbf{u}(t), \mathbf{d}(t))$ previously defined and the function

$$G(\mathbf{u}(t), \mathbf{d}(t)) = \nu|\nabla \mathbf{u}(t)|_2^2 + |-\Delta \mathbf{d}(t) + f(\mathbf{d}(t))|_2^2$$

and check that the following differential inequalities hold for all $t \in (0, +\infty)$:

$$\frac{d}{dt}E(\mathbf{u}(t), \mathbf{d}(t)) + G(\mathbf{u}(t), \mathbf{d}(t)) \leq 0 \quad \text{and} \quad \frac{d}{dt}G(\mathbf{u}(t), \mathbf{d}(t)) \leq C(G(\mathbf{u}(t), \mathbf{d}(t))^3 + 1).$$

Then, one can prove the convergence for $E(\mathbf{u}(t), \mathbf{d}(t))$, $\mathbf{u}(t)$, $-\Delta \mathbf{d}(t) + f(\mathbf{d}(t))$ when $t \rightarrow \infty$ and for \mathbf{d} only weakly in \mathbf{H}^2 by subsequences (i.e. $\forall (t_n) \uparrow +\infty$, there exists a subsequence (t_{nk}) and a critical point $\bar{\mathbf{d}}$ such that $\mathbf{d}(t_{nk}) \rightharpoonup \bar{\mathbf{d}}$ in \mathbf{H}^2). Thus, it lacks to prove the uniqueness of limit of $\mathbf{d}(t)$ because the set of critical points might be a continuum.

Step 2: By using the following **Łojasiewicz-Simon type inequality** [Wu'10]:

Let $\bar{\mathbf{d}}$ be a critical point of $E_e(\mathbf{d})$ subject to $\mathbf{d} \in \mathbf{H}^1(\Omega)$ with the boundary condition $\mathbf{d}|_{\partial\Omega} = \mathbf{h}$. There exist constants $\theta \in (0, 1/2)$ and $\beta > 0$ depending on $\bar{\mathbf{d}}$ such that for any $\mathbf{d} \in \mathbf{H}^1(\Omega)$ satisfying $\mathbf{d}|_{\partial\Omega} = \mathbf{h}$ and $\|\mathbf{d} - \bar{\mathbf{d}}\|_1 < \beta$, there holds

$$\|-\Delta \mathbf{d} + f(\mathbf{d})\|_{-1} \geq |E_e(\mathbf{d}) - E_e(\bar{\mathbf{d}})|^{1-\theta},$$

one can obtain that for all t such that $\|\mathbf{d}(t) - \bar{\mathbf{d}}\|_1 < \beta$, the following differential inequality holds:

$$\frac{C}{\theta} \frac{d}{dt} ((E(\mathbf{u}(t), \mathbf{d}(t)) - E_e(\bar{\mathbf{d}}))^\theta) + G(\mathbf{u}(t), \mathbf{d}(t))^{1/2} \leq 0. \quad (29)$$

Step 3: By an argument of contradiction, one deduces that there exists n_0 big enough such that $\|\mathbf{d}(t) - \bar{\mathbf{d}}\|_1 < \beta$ for all $t \geq t_{n_0}$

Step 4: From (29) for all $t \geq t_{n_0}$ one gets

$$\int_{t_{n_0}}^{+\infty} G(\mathbf{u}(t), \mathbf{d}(t))^{1/2} \leq \frac{C}{\theta} (E(\mathbf{u}(t_{n_0}), \mathbf{d}(t_{n_0})) - E_e(\bar{\mathbf{d}}))^\theta \leq C.$$

Hence, in particular, $\int_{t_{n_0}}^{+\infty} |\partial_t \mathbf{d}|_2 \leq C$. This last bound implies that $(\mathbf{d}(t))_{t \geq t_{n_0}}$ is a Cauchy sequence in $\mathbf{L}^2(\Omega)$ as $t \uparrow +\infty$, hence $\mathbf{d}(t) \rightarrow \bar{\mathbf{d}}$ in $\mathbf{L}^2(\Omega)$. Finally, this strong convergence also can be proved in $\mathbf{H}^2(\Omega)$

Remark: The argument of Step 1 is done in [Climent et al.'10] for a model with stretching terms and periodic boundary conditions for \mathbf{d} . The arguments of Step 2, 3 and 4 are done in [Liu, Wu, Xu'09] (for periodic boundary conditions) and [Wu'10] (for Dirichlet boundary conditions).

Theorem 7 (Stability) *Under conditions of Theorem 6, for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if*

$$E(\mathbf{u}_0, \mathbf{d}_0) - E_\infty \leq \delta(\varepsilon) \quad (30)$$

and

$$G(\mathbf{u}_0, \mathbf{d}_0) := \nu |\nabla \mathbf{u}_0|_2^2 + |\Delta \mathbf{d}_0 - \mathbf{f}(\mathbf{d}_0)|_2^2 \leq \frac{\varepsilon}{3}, \quad (31)$$

then for each $t \geq 0$, one has:

$$G(\mathbf{u}(t), \mathbf{d}(t)) := \nu |\nabla \mathbf{u}(t)|_2^2 + |\Delta \mathbf{d}(t) - \mathbf{f}(\mathbf{d}(t))|_2^2 \leq \varepsilon.$$

Remark: Hypothesis (31) means that $(\mathbf{u}_0, \mathbf{d}_0)$ is near to an equilibrium state $(0, \mathbf{d}_*)$ and (30) means that the total decay of the energy is small enough. Arguing as in [Climent et al.'10], it is not difficult to prove that hypothesis (30) implies in particular the global regularity in $(0, +\infty)$ without imposing large viscosity ν . Moreover, this hypothesis (30) includes the

particular case where the initial data $(\mathbf{u}_0, \mathbf{d}_0)$ is near to a global minimizer $(0, \mathbf{d}_\star)$ (because $E_\infty = E(0, \mathbf{d}_\star)$), where the global regularity can also be proved without imposing large viscosity ν , see [Lin,Liu'95]. In the recent paper [Petzeltova et al.], there are some more specific stability results than in Theorem 7, for instance, changing hypothesis (30) by the more general assumption that \mathbf{d}_0 is near to a local minimizer of the elastic energy $E_e(\mathbf{d})$ (and \mathbf{u}_0 near of zero).

2.3 Time-periodic problem

Let $T > 0$ a finite fixed number which states the time period, and a boundary data $\mathbf{h}(t)$ for $\mathbf{d}(t)$ time-dependent and time-periodic, i.e. $\mathbf{h}(0) = \mathbf{h}(T)$.

Definition 8 (\mathbf{u}, \mathbf{d}) is said a weak time-periodic solution in $(0, T)$ of (15), (8) and (12) if

$$\mathbf{u} \in L^\infty(0, T; \mathbf{H}) \cap L^2(0, T; \mathbf{V}), \quad \mathbf{d} \in L^\infty(0, T; \mathbf{H}^1) \cap L^2(0, T; \mathbf{H}^2)$$

satisfying (15) and boundary conditions (8) as in Definition 1 and time-periodic conditions $\mathbf{u}(0) = \mathbf{u}(T)$, $\mathbf{d}(0) = \mathbf{d}(T)$ in the sense of spaces \mathbf{L}^2 and \mathbf{H}^1 respectively.

Definition 9 A weak time-periodic solution in $(0, T)$ of (15), (8) and (12) is said a strong solution if

$$\mathbf{u} \in L^\infty(0, T; \mathbf{H}^1) \cap L^2(0, T; \mathbf{H}^2), \quad \mathbf{d} \in L^\infty(0, T; \mathbf{H}^2) \cap L^2(0, T; \mathbf{H}^3)$$

and verifying point-wise the fully differential system (15).

Theorem 10 (Existence of weak time-periodic solutions) Let Ω and \mathbf{h} be regular enough with $\mathbf{h}(0) = \mathbf{h}(T)$ on $\partial\Omega$ and such that the steady lifting function $\tilde{\mathbf{d}}$ defined in (19) satisfies $\tilde{\mathbf{d}} \in L^\infty(0, T; \mathbf{H}^1)$. Then, there exists a weak time periodic solution (\mathbf{u}, \mathbf{d}) of problem (15), (8) and (12).

To prove this theorem, a fully Galerkin discretization is introduced (approximating in finite dimension both variables \mathbf{u} and \mathbf{d}), proving existence and uniqueness of approximate solution associated to arbitrary initial conditions. The finite-dimensional Galerkin problem let us find time-periodic approximate solutions via a fixed-point argument (Leray-Schauder's Theorem) applied to the operator mapping the initial and final time values. This allows us to obtain a time-periodic Galerkin solution, which converges towards a time-periodic solution of the continuous problem. Reasoning in the same way that in Theorem 5.2 of [Climent, Guillen'10], also in this case, the use of the Maximum Principle is not necessary.

Theorem 11 (Existence of strong time-periodic solutions for ν large) *Under conditions of Theorem 10, if moreover $\tilde{\mathbf{d}} \in L^\infty(0, T; \mathbf{H}^3)$ and $\partial_t \tilde{\mathbf{d}} \in L^\infty(0, T; \mathbf{H}^1)$ then, for each $\nu \geq \nu_0$ for a certain $\nu_0 = \nu_0(T, \partial_t \tilde{\varphi})$, there exists a strong time-periodic solution of (15), (8) and (12).*

To prove this theorem it suffices to use the existence of weak time-periodic solutions, and for the initial-valued problem, the weak/strong uniqueness and the existence of global strong solution for big enough viscosity ν (see [Climent, Guillen'10]).

3 Smectic-A Problem

The initial-boundary value problem (16) with time-independent boundary data has been studied in [Liu'00] obtaining existence of weak solutions in $[0, T]$ for all $T > 0$, existence of regular solution for big enough viscosity and uniqueness of weak/regular solution. Here, we are going to show some results concerning time-dependent boundary data that are developed in [Climent, Guillen'10], although some of them (as the proof of Theorem 16) are slightly simplified.

We define the lifting function $\tilde{\varphi} = \tilde{\varphi}(t)$ as the weak solution of the problem

$$\begin{cases} \Delta^2 \tilde{\varphi} = 0 & \text{in } \Omega, \\ \tilde{\varphi} = \varphi_1(t), \quad \partial_n \tilde{\varphi} = \varphi_2(t) & \text{on } \partial\Omega. \end{cases} \quad (32)$$

In the time-periodic case, since by hypothesis $\varphi_1(0) = \varphi_1(T)$ and $\varphi_2(0) = \varphi_2(T)$ on $\partial\Omega$, then $\tilde{\varphi}(0) = \tilde{\varphi}(T)$ in Ω .

If we define $\hat{\varphi}(t) = \varphi(t) - \tilde{\varphi}(t)$, then $\Delta^2 \hat{\varphi} = \Delta^2 \varphi$ in Q and $\hat{\varphi} = \nabla \hat{\varphi} = 0$ on Σ . In the time-periodic case, one has $\varphi(0) = \varphi(T)$ if and only if $\hat{\varphi}(0) = \hat{\varphi}(T)$. Then, we can rewrite the problem (16) respect to the variables $(\mathbf{u}, \hat{\varphi})$ (with $\hat{\varphi}(t) = \varphi(t) - \tilde{\varphi}(t)$) as follows (recall that all coefficients have been taken equal to one, excepting viscosity $\nu = \mu_4/2$):

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} - \nabla \cdot \sigma_{nl}^d - (\Delta^2 \hat{\varphi} - \nabla \cdot \mathbf{f}(\nabla \varphi)) \nabla \varphi + \nabla p = 0, \\ \nabla \cdot \mathbf{u} = 0, \\ \partial_t \hat{\varphi} + \mathbf{u} \cdot \nabla \varphi + \Delta^2 \hat{\varphi} - \nabla \cdot \mathbf{f}(\nabla \varphi) = \partial_t \tilde{\varphi}, \\ \mathbf{u}|_{\partial\Omega} = 0, \quad \hat{\varphi}|_{\partial\Omega} = 0, \quad \partial_n \hat{\varphi}|_{\partial\Omega} = 0 \quad \text{on } \Sigma_T \end{cases} \quad (33)$$

jointly with either initial conditions $\mathbf{u}(0) = \mathbf{u}_0$, $\hat{\varphi}(0) = \varphi_0 - \tilde{\varphi}(0)$ or time-periodic conditions $\mathbf{u}(0) = \mathbf{u}(T)$, $\hat{\varphi}(0) = \hat{\varphi}(T)$.

If (\mathbf{u}, φ) is a regular enough solution of (33), the following energy equality holds:

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \|\mathbf{u}\|_2^2 + \frac{1}{2} \|\Delta \hat{\varphi}\|_2^2 + \int_{\Omega} F(\nabla \varphi) \right) + |\nabla \varphi^T D(\mathbf{u}) \nabla \varphi|_2^2 + |D(\mathbf{u}) \nabla \varphi|_2^2 \\ & + \nu \|\nabla \mathbf{u}\|_2^2 + \|\Delta^2 \hat{\varphi} - \nabla \cdot \mathbf{f}(\nabla \varphi)\|_2^2 = (\partial_t \tilde{\varphi}, \Delta^2 \hat{\varphi} - \nabla \cdot \mathbf{f}(\nabla \varphi)) + (\partial_t \nabla \tilde{\varphi}, \mathbf{f}(\nabla \varphi)) \end{aligned} \quad (34)$$

for $t \in (0, +\infty)$. Here, the auxiliary variable $w = \Delta^2 \hat{\varphi} - \nabla \cdot \mathbf{f}(\nabla \varphi)$ has been used. To obtain the strong estimates (see (38), (39) below), we consider an other auxiliary variable $\hat{w} := w - \partial_t \tilde{\varphi}$ (because $\hat{w}|_{\partial\Omega} = 0$), $\tilde{\varphi}$ regular enough and the weak estimate $\varphi \in L^\infty(0, +\infty; H^2)$, then, we can deduce

$$\|\hat{\varphi}\|_4 \leq C(|\hat{w}|_2 + 1), \quad \|\varphi\|_4 \leq C(|\hat{w}|_2 + 1),$$

and, for $t \in (0, +\infty)$:

$$\begin{aligned} & \frac{d}{dt} (\|\mathbf{u}\|_1^2 + |\hat{w}|_2^2) + \frac{\nu}{2} \|\mathbf{u}\|_2^2 + \|\hat{w}\|_2^2 \leq C(|\hat{w}|_2^2 + 1) \\ & + \frac{C}{\nu} \left(\|\hat{w}\|_2^2 (1 + |\hat{w}|_2 + |\hat{w}|_2^2) + \|\mathbf{u}\|_2^2 (\|\mathbf{u}\|_1^2 + |\hat{w}|_2^2 + 1) \right). \end{aligned} \quad (35)$$

Inequality (35) is slight different from the inequality stated in [Climent, Guillen'10] because in that paper the auxiliary variable $\hat{w} = \partial_t \hat{\varphi} + \mathbf{u} \cdot \nabla \hat{\varphi} = -\Delta^2 \hat{\varphi} + \nabla \cdot \mathbf{f}(\nabla \varphi) + \partial_t \tilde{\varphi} - \mathbf{u} \cdot \nabla \tilde{\varphi}$ was used.

3.1 The initial value problem up to infinite time

Definition 12 We say that (\mathbf{u}, φ) is a weak solution of (16)-(11) in $(0, +\infty)$ if

$$\nabla \cdot \mathbf{u} = 0 \text{ in } Q, \quad \mathbf{u}|_{\Sigma} = 0, \quad \varphi|_{\Sigma} = \varphi_1, \quad \partial_n \varphi|_{\Sigma} = \varphi_2 \quad \text{a.e. } t \in (0, +\infty),$$

$$\|\mathbf{u}(t), \varphi(t)\|_{0 \times 2} \leq C_1 \quad \forall t \geq 0 \quad (36)$$

$$\forall \gamma > 0, \quad e^{-\gamma t} \int_0^t e^{\gamma s} \|\mathbf{u}(s), \varphi(s)\|_{1 \times 4}^2 ds \leq C_2 \left(1 + \frac{1}{\nu} \right), \quad \forall t \geq 0, \quad (37)$$

where $C_1, C_2 > 0$ are constants independent of ν , verifying

$$\begin{aligned} & \langle \partial_t \mathbf{u}, \mathbf{v} \rangle + ((\mathbf{u} \cdot \nabla) \mathbf{u}, \mathbf{v}) + \nu (\nabla \mathbf{u}, \nabla \mathbf{v}) + (\sigma_{nl}^d, \nabla \mathbf{v}) \\ & - ((\Delta^2 \varphi - \nabla \cdot \mathbf{f}(\nabla \varphi)) \nabla \varphi, \mathbf{v}) = 0 \quad \text{in } \mathcal{D}'(0, +\infty), \quad \forall \mathbf{v} \in \mathbf{V}, \\ & \partial_t \varphi + (\mathbf{u} \cdot \nabla) \varphi + \Delta^2 \varphi - \nabla \cdot \mathbf{f}(\nabla \varphi) = 0, \quad \text{a.e. in } (0, +\infty) \times \Omega \\ & \mathbf{u}(0) = \mathbf{u}_0, \quad \varphi(0) = \varphi_0 \quad \text{in } \Omega. \end{aligned}$$

In the finite time case ($T < \infty$), (37) holds even when $\gamma = 0$, i.e. $(\mathbf{u}, \varphi) \in L^2(0, T; \mathbf{H}^1 \times H^4)$.

Definition 13 We say that a weak solution (\mathbf{u}, φ) of (16)-(11) is a strong solution if

$$\|\mathbf{u}(t), \varphi(t)\|_{1 \times 4} \leq C_3 \quad \forall t \geq 0, \quad (38)$$

$$\forall \gamma > 0, \quad e^{-\gamma t} \int_0^t e^{\gamma s} \|\mathbf{u}(s), \varphi(s)\|_{2 \times 6}^2 ds \leq C_4, \quad \forall t \geq 0 \quad (39)$$

and verifying point-wise the fully differential system (16).

Now, we state three results given in [Climent,Guillen'10].

Theorem 14 (weak/strong uniqueness) *If $(\mathbf{u}_1, \varphi_1)$ and $(\mathbf{u}_2, \varphi_2)$ are respectively a weak and a strong solution of (16)-(11), then $\mathbf{u}_1 = \mathbf{u}_2$ and $\varphi_1 = \varphi_2$.*

Theorem 15 (Existence of weak solutions) *Let $\mathbf{u}_0 \in \mathbf{H}$ and $\varphi_0 \in H^2$. Let Ω , φ_1 and φ_2 be regular enough, verifying the compatibility conditions $\varphi_0|_{\partial\Omega} = \varphi_1(0)$, $\partial_n \varphi_0|_{\partial\Omega} = \varphi_2(0)$ and such that the lifting function $\tilde{\varphi}$ defined in (32) satisfies*

$$\tilde{\varphi} \in L^\infty(0, +\infty; \mathbf{H}^4(\Omega)) \quad \text{and} \quad \partial_t \tilde{\varphi} \in L^\infty(0, +\infty; \mathbf{W}^{1,4}(\Omega)).$$

Then, there exists a weak solution (\mathbf{u}, φ) of (16)-(11) in $(0, +\infty)$.

The proof is based on a semi-Galerkin method as in [Liu'00]. The novelty respect to [Liu'00] is that in [Climent,Guillen'10] we will find a weak solution bounded up to infinity time, even imposing time-dependent boundary conditions for the layer variable φ .

Theorem 16 (Existence of strong solutions for ν large) *In the conditions of Theorem 15, if moreover $(\mathbf{u}_0, \varphi_0) \in \mathbf{H}^1 \times H^4$ with $\|\mathbf{u}_0\|_1 \leq R_1$, $\|\varphi_0\|_4 \leq R_2$,*

$$\partial_t \tilde{\varphi} \in L^\infty(0, +\infty; \mathbf{W}^{1,4}(\Omega)) \quad \text{and} \quad \partial_{tt} \tilde{\varphi} \in L^\infty(0, +\infty; \mathbf{L}^2(\Omega)),$$

then there exists $\nu_0 = \nu_0(R_1, R_2, \partial_t \tilde{\varphi}, \partial_{tt} \tilde{\varphi})$ such that for each $\nu \geq \nu_0$, there exists a unique strong solution of (16)-(11) in $(0, +\infty)$, which satisfies (38) and (39) with constants C_3 and C_4 depending on ν_0 (but independent of ν).

The proof of this theorem is based in the following inequality obtained from (35):

$$\begin{aligned} & \frac{d}{dt}(\Phi_1 + \Phi_2) + \left(\frac{\nu}{2} - \frac{C}{\nu}(\Phi_1 + \Phi_2 + 1) \right) \Psi_1 \\ & + \left(\frac{1}{2} - \frac{C}{\nu}(1 + \Phi_2^{1/2} + \Phi_2) \right) \Psi_2 \leq C(\Phi_2 + 1) \end{aligned} \quad (40)$$

for $t \in (0, +\infty)$, where

$$\Phi_1(t) = \|\mathbf{u}\|_1^2, \quad \Phi_2(t) = |\hat{w}|_2^2, \quad \Psi_1(t) = \|\mathbf{u}\|_2^2, \quad \Psi_2(t) = \|\hat{w}\|_2^2.$$

Again, inequality (40) is rather similar to inequality obtained in [Climent,Guillen'10].

3.2 Behavior at infinite time

In this section, we assume time-independent boundary data. We will present two results for the Smectic-A case corresponding with Theorems 6 and 7 for the Nematic case. In fact, the proof follows the same lines as in the nematic case.

Theorem 17 (Asymptotic stability) *Under conditions of Theorem 16 the energy*

$$E(\mathbf{u}(t), \varphi(t)) = E_k(\mathbf{u}(t)) + E_e(\varphi(t)) = \frac{1}{2}|\nabla \mathbf{u}(t)|^2 + \int_{\Omega} \left(\frac{1}{2}|\Delta \varphi|^2 + F(\nabla \varphi) \right)$$

(sum of kinetic and elastic energies), satisfies

$$E(\mathbf{u}(t), \varphi(t)) \searrow E_{\infty}$$

when $t \uparrow +\infty$ and the strong solution (\mathbf{u}, φ) satisfies

$$\mathbf{u}(t) \rightarrow 0 \text{ in } \mathbf{H}_0^1(\Omega), \quad (\Delta^2 \varphi - \nabla \cdot f(\nabla \varphi))(t) \rightarrow 0 \text{ in } L^2(\Omega)$$

when $t \uparrow +\infty$.

Moreover, for each sequence $t_j \rightarrow +\infty$, there exists a subsequence $t_{jk} \rightarrow +\infty$ such that $\varphi(t_{jk}) \rightarrow \bar{\varphi}$ in $H^4(\Omega)$ -weak, where $\bar{\varphi}$ is a critical point of the elastic energy, that is, a solution of the stationary problem

$$\begin{cases} \Delta^2 \bar{\varphi} - \nabla \cdot f(\nabla \bar{\varphi}) = 0 & \text{in } \Omega, \\ \bar{\varphi}|_{\partial\Omega} = \varphi_1, \quad \partial_n \bar{\varphi}|_{\partial\Omega} = \varphi_2 \end{cases}$$

and any possible critical point limit $\bar{\varphi}$ must have the same elastic energy equal to the limit of the total energy, that is, $E_e(\bar{\varphi}) = E_{\infty}$

Remark: In smectic-A case, the uniqueness of the critical point $\bar{\varphi}$ (as limit of the trajectory at infinity time) remains open. This problem is considered in a submitted paper [Segatti, Wu'10], where a specific Łojasiewicz-Simon inequality is proved giving a relation between the residual $\Delta^2 \varphi - \nabla \cdot f(\nabla \varphi)$ and the energies $E_e(\varphi) - E_e(\bar{\varphi})$, for any φ near from $\bar{\varphi}$.

Theorem 18 (Stability) *In the conditions of theorem 17, $\forall \varepsilon > 0$, there exists $\delta(\varepsilon)$ such that if $E(u_0, \varphi_0) - E_{\infty} \leq \delta(\varepsilon)$ and $\nu|\nabla \mathbf{u}_0|_2^2 + |\Delta^2 \varphi_0 - \nabla \cdot \mathbf{f}(\nabla \varphi_0)|_2^2 \leq \frac{\varepsilon}{3}$, then for each $t \geq 0$, one has:*

$$\nu|\nabla \mathbf{u}(t)|_2^2 + |\Delta^2 \varphi(t) - \nabla \cdot \mathbf{f}(\nabla \varphi(t))|_2^2 \leq \varepsilon.$$

3.3 The time-periodic problem

Let $T > 0$ a finite fixed number and a boundary data for φ time-dependent and time-periodic, i.e. $\varphi_1(0) = \varphi_1(T)$ and $\varphi_2(0) = \varphi_2(T)$. The results of this section can be found in [Climent, Guillen'10].

Definition 19 *We say that (\mathbf{u}, φ) is a weak time-periodic solution of (16), (9) and (13) if*

$$\mathbf{u} \in L^\infty(0, T; \mathbf{H}) \cap L^2(0, T; \mathbf{H}^1), \quad \varphi \in L^\infty(0, T; \mathbf{H}^2) \cap L^2(0, T; \mathbf{H}^4)$$

satisfying (16) and boundary conditions (9) as in Definition 12 and time-periodic conditions $\mathbf{u}(0) = \mathbf{u}(T)$, $\varphi(0) = \varphi(T)$ in the sense of spaces \mathbf{L}^2 and \mathbf{H}^2 respectively.

Definition 20 *We say that a weak time periodic solution of (16), (9) and (13) is a strong solution if*

$$\mathbf{u} \in L^\infty(0, T; \mathbf{H}^1) \cap L^2(0, T; \mathbf{H}^2), \quad \varphi \in L^\infty(0, T; \mathbf{H}^4) \cap L^2(0, T; \mathbf{H}^6)$$

and verifying point-wise the fully differential system (16).

We only present the two main results of existence of (weak and strong) time-periodic solutions.

Theorem 21 (Existence of weak time-periodic solutions) *Let Ω , φ_1 and φ_2 be regular enough with $\varphi_1(0) = \varphi_1(T)$, $\varphi_2(0) = \varphi_2(T)$, and such that the lifting function $\tilde{\varphi}$ defined in (32) satisfies*

$$\tilde{\varphi} \in L^\infty(0, T; \mathbf{H}^4(\Omega)), \quad \partial_t \tilde{\varphi} \in L^\infty(0, T; \mathbf{W}^{1,4}(\Omega)).$$

Then, there exists a weak time-periodic solution of (16), (9) and (13).

Theorem 22 (Existence of regular time-periodic solutions for ν large) *Under conditions of previous theorem, if moreover*

$$\partial_t \tilde{\varphi} \in L^\infty(0, +\infty; \mathbf{W}^{1,4}(\Omega)) \quad \text{and} \quad \partial_{tt} \tilde{\varphi} \in L^\infty(0, +\infty; \mathbf{L}^2(\Omega)),$$

then there exists $\nu_0 = \nu_0(\partial_t \tilde{\varphi}, \partial_{tt} \tilde{\varphi})$ such that, for each $\nu \geq \nu_0$, there exists a strong time-periodic solution of (16), (9) and (13).

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